

Periodic Jacobi operator with finitely supported perturbations: the inverse resonance problem.

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Abstract

We consider a periodic Jacobi operator H with finitely supported perturbations on \mathbb{Z} . We solve the inverse resonance problem: we prove that the mapping from finitely supported perturbations to the scattering data, the inverse of the transmission coefficient and the Jost function on the right half-axis, is one-to-one and onto. We consider the problem of reconstruction of the scattering data from all eigenvalues, resonances and the set of zeros of $R_-(\lambda) + 1$, where R_- is the reflection coefficient.

Keywords: resonances, inverse scattering, Jacobi operator, periodic

1 Introduction.

We consider a Jacobi operator $H = H^0 + V$ on the lattice $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Here the unperturbed operator H^0 is a periodic Jacobi operator given by

$$(H^0 y)_n = a_{n-1}^0 y_{n-1} + a_n^0 y_{n+1} + b_n^0 y_n, \quad (1.1)$$

where $y = (y_n)_1^\infty \in \ell^2 = \ell^2(\mathbb{Z})$ and the q -periodic coefficients $a_n^0, b_n^0 \in \mathbb{R}$ satisfy

$$a_n^0 = a_{n+q}^0 > 0, \quad b_n^0 = b_{n+q}^0, \quad n \in \mathbb{Z}, \quad \prod_{j=1}^q a_j^0 = 1, \quad q \geq 2. \quad (1.2)$$

We fix a positive integer p . The perturbation operator V is the finitely supported Jacobi operator given by

$$(Vy)_n = \begin{cases} u_{n-1}y_{n-1} + u_n y_{n+1} + v_n y_n, & \text{if } 1 \leq n \leq p, \\ u_p y_p, & \text{if } n = p+1, \\ u_0 y_1 + v_0 y_0, & \text{if } n = 0, \\ 0, & \text{if } n \leq -1 \text{ or } n \geq p+2, \quad p \geq 1. \end{cases} \quad (1.3)$$

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We parameterize V by the vector $(u, v) \in \mathbb{R}^{2p}$ and let (u, v) belong to the class \mathcal{V}_ν given by

$$\mathcal{V}_\nu = \left\{ (u, v) \in \mathbb{R}^{2p} : a_n^0 + u_n > 0, \ n = 0, \dots, p, \ u_p \neq 0, \ v_0 \neq 0 \right\} \text{ if } \nu = 2p, \quad (1.4)$$

$$\mathcal{V}_\nu = \left\{ (u, v) \in \mathbb{R}^{2p} : a_n^0 + u_n > 0, \ n = 0, \dots, p, \ u_p = 0, \ v_0 \neq 0, \ v_p \neq 0, \right\} \text{ if } \nu = 2p - 1. \quad (1.5)$$

We rewrite H in the form

$$(Hy)_n = a_{n-1}y_{n-1} + a_n y_{n+1} + b_n y_n \quad (1.6)$$

with the coefficients a_n, b_n given by

$$a_n = \begin{cases} a_n^0 + u_n > 0 & \text{if } 0 \leq n \leq p, \\ a_n^0 & \text{if } n \leq -1 \text{ or } n \geq p+1, \end{cases} \quad b_n = \begin{cases} b_n^0 + v_n & \text{if } 0 \leq n \leq p, \\ b_n^0 & \text{if } n \leq -1 \text{ or } n \geq p+1. \end{cases} \quad (1.7)$$

The corresponding Jacobi matrices have the forms

$$H^0 = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & a_0^0 & b_1^0 & a_1^0 & 0 & 0 & \dots \\ \dots & 0 & a_1^0 & b_2^0 & a_2^0 & 0 & \dots \\ \dots & 0 & 0 & a_2^0 & b_3^0 & b_3^0 & \dots \\ \dots & 0 & 0 & 0 & a_3^0 & b_4^0 & \dots \\ \dots & 0 & 0 & 0 & 0 & a_4^0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad H = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & a_0 & b_1 & a_1 & 0 & 0 & \dots \\ \dots & 0 & a_1 & b_2 & a_2 & 0 & \dots \\ \dots & 0 & 0 & a_2 & b_3 & b_3 & \dots \\ \dots & 0 & 0 & 0 & a_3 & b_4 & \dots \\ \dots & 0 & 0 & 0 & 0 & a_4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (1.8)$$

For $a_n = 1, b_n^0 = 0, n \in \mathbb{Z}$, the operator H is the finite difference Schrödinger operator with finitely supported potential.

A lot of papers is devoted to the direct and inverse resonance problems for the Schrödinger operator $-\frac{d^2}{dx^2} + q(x)$ on the line \mathbb{R} with compactly supported perturbation (see [S], [Fr], [Z], [K3] and references given there). Zworski [Z] obtained the first results about the distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line. One of the present authors obtained the uniqueness, the recovery and the characterization of the S -matrix for the Schrödinger operator with a compactly supported potential on the real line [K3], see also [Z1], [BKW] concerning the uniqueness.

The problem of resonances for the Schrödinger with periodic plus compactly supported potential $-\frac{d^2}{dx^2} + p(x) + q(x)$ is much less studied: [F1], [KM], [K1]. The following results were obtained in [K1]: 1) the distribution of resonances in the disk with large radius is determined, 2) some inverse resonance problem, 3) the existence of a logarithmic resonance-free region near the real axis. The inverse resonance problem is not yet solved.

Finite-difference Schrödinger and Jacobi operators express many similar features. Spectral and scattering properties of infinite Jacobi matrices are much studied (see [Mo], [DS1],

[DS2] and references given there). The inverse problem was solved for periodic Jacobi operators: [P].

The inverse scattering problem for asymptotically periodic coefficients was solved by Khanmamedov: [Kh1] (note that the russian versions were dated much earlier) and Egorova, Michor and Teschl [EMT] (in the case of quasi-periodic background).

The resonance problems are less studied (see M.Marletta and R.Weikard [MW]). The inverse resonances problem was recently solved in the case of constant background [K2].

In [IK1] we consider the direct resonance problem in the case of periodic background. We describe the spectral and scattering properties of H . Moreover, in the special case $u_n \equiv 0$ we obtain the asymptotics of the spectrum in the limit of small perturbations V . In Theorem 1.1 below we summarize some results obtained in [IK1].

In [IK2] we consider the zigzag half-nanotubes (tight-binding approximation) in a uniform magnetic field which is described by the magnetic Schrödinger operator with a periodic potential plus a finitely supported perturbation on the half-lattice. We describe all eigenvalues and resonances of this operator, and their dependence on the magnetic field.

In [IK3] we consider a periodic Jacobi operator with finitely supported perturbations on the half-lattice. We describe all eigenvalues and resonances, and give their properties. We solve the inverse resonance problem: we prove that the mapping from finitely supported perturbations to the Jost functions is one-to-one and onto, we show how the Jost functions can be reconstructed from all eigenvalues, resonances and from the set of zeros of $S(\lambda) - 1$, where $S(\lambda)$ is the scattering matrix.

In the present paper we extend the methods from [IK3] to the inverse resonance problem on the lattice \mathbb{Z} . In one aspect the inverse scattering problem for the perturbed operator H on \mathbb{Z} is simpler then on the half-lattice: even the unperturbed periodic Jacobi operator on the half-lattice has bound and antibound states. But technically the inverse problem on the lattice is more involved as we need to reconstruct two analytic functions on the two-sheeted Riemann surface: the numerator and denominator of the reflection coefficient R_- , instead of one as in [IK3], the Jost function f_0^+ .

Now we pass to the description of the spectral and scattering properties of H , recalling some results from [IK1], and formulate our main results.

The spectrum of H^0 on $\ell^2(\mathbb{Z})$ is absolutely continuous and consists of q zones σ_j separated by the gaps γ_j given by

$$\begin{aligned} \sigma_j &= [\lambda_{j-1}^+, \lambda_j^-], \quad j = 1, \dots, q, \quad \gamma_j = (\lambda_j^-, \lambda_j^+), \quad j = 1, \dots, q-1, \\ \lambda_0^+ &< \lambda_1^- \leq \lambda_1^+ < \dots < \lambda_{q-1}^- \leq \lambda_{q-1}^+ < \lambda_q^-. \end{aligned} \quad (1.9)$$

We denote $\gamma_0 = (-\infty, \lambda_0^+)$ and $\gamma_q = (\lambda_q^+, +\infty)$ the infinite gaps.

Let $\varphi = (\varphi_n(\lambda))_1^\infty$ and $\vartheta = (\vartheta_n(\lambda))_1^\infty$ be fundamental solutions for the equation

$$a_{n-1}^0 y_{n-1} + a_n^0 y_{n+1} + b_n^0 y_n = \lambda y_n, \quad \lambda \in \mathbb{C}, \quad (1.10)$$

satisfying the conditions $\vartheta_0 = \varphi_1 = 1$ and $\vartheta_1 = \varphi_0 = 0$. Here and below $a_0^0 = a_q^0$. Introduce the Lyapunov function Δ by

$$\Delta = \frac{\varphi_{q+1} + \vartheta_q}{2}. \quad (1.11)$$

It is known that $\Delta(\lambda)$ is a polynomial of degree q and $\lambda_j^\pm, j = 1, \dots, q$, are the zeros of the polynomial $\Delta^2(\lambda) - 1$ of degree $2q$. Note that $\Delta(\lambda_j^\pm) = (-1)^{q-j}$. In each “gap” $[\lambda_j^-, \lambda_j^+]$ there is one simple zero of polynomials $\varphi_q, \dot{\Delta}, \vartheta_{q+1}$. Here and below \dot{f} denotes the derivative of $f = f(\lambda)$ with respect to λ : $\dot{f} \equiv \partial_\lambda f \equiv f'(\lambda)$.

Let Γ denote the complex plane cut along the segments σ_j (1.9): $\Gamma = \mathbb{C} \setminus \sigma_{ac}(H^0)$. Now we introduce the two-sheeted Riemann surface Λ of $\sqrt{1 - \Delta^2(\lambda)}$ by joining the upper and lower rims of two copies of the cut plane Γ in the usual (crosswise) way. We identify the first (physical) sheet Λ_1 with Γ and the second sheet we denote by Λ_2 .

Let \sim denote the natural projection from Λ into the complex plane:

$$\lambda \in \Lambda, \quad \lambda \rightarrow \tilde{\lambda} \in \mathbb{C}. \quad (1.12)$$

By identification of $\Gamma = \mathbb{C} \setminus \sigma_{ac}(H^0)$ with Λ_1 , map \sim can be also considered to be projection from Λ into the physical sheet Λ_1 .

The j -th gap on the first physical sheet Λ_1 we will denote by γ_j^+ and the same gap but on the second nonphysical sheet Λ_2 we will denote by γ_j^- and let γ_j^c be the union of $\overline{\gamma_j^+}$ and $\overline{\gamma_j^-}$:

$$\gamma_j^c = \overline{\gamma_j^+} \cup \overline{\gamma_j^-}. \quad (1.13)$$

Define the function $\Omega(\lambda) = \sqrt{1 - \Delta^2(\lambda)}$, $\lambda \in \Lambda$, by

$$\Omega(\lambda) < 0 \quad \text{for} \quad \lambda \in (\lambda_{q-1}^+, \lambda_q^-) \subset \Lambda_1. \quad (1.14)$$

Introduce the Bloch functions ψ_n^\pm and the Titchmarsh-Weyl functions m_\pm on Λ by

$$\psi_n^\pm(\lambda) = \vartheta_n(\lambda) + m_\pm(\lambda)\varphi_n(\lambda), \quad (1.15)$$

$$m_\pm(\lambda) = \frac{\phi(\lambda) \pm i\Omega(\lambda)}{\varphi_q}, \quad \phi = \frac{\varphi_{q+1} - \vartheta_q}{2}, \quad \lambda \in \Lambda_1. \quad (1.16)$$

The projection of all singularities of m_\pm to the complex plane coincides with the set of zeros $\{\mu_j\}_{j=1}^{q-1}$ of polynomial φ_q . Recall that $\vartheta_n, \varphi_n, \phi$ are polynomials. Recall that any polynomial $P(\lambda)$ gives rise to a function $P(\lambda) = P(\tilde{\lambda})$ on the Riemann surface Λ of $\sqrt{1 - \Delta^2(\lambda)}$.

The perturbation V satisfying (1.3) does not change the absolutely continuous spectrum:

$$\sigma_{ac}(H) = \sigma_{ac}(H^0) = \bigcup_{n=1}^q [\lambda_{n-1}^+, \lambda_n^-]. \quad (1.17)$$

The spectrum of H consists of an absolutely continuous part $\sigma_{ac}(H) = \sigma_{ac}(H^0)$ plus a finite number of simple eigenvalues in each non-empty gap $\gamma_n, n = 0, \dots, q$.

Introduce the function

$$\alpha(\lambda) = C \det((H - \lambda)(H^0 - \lambda)^{-1}) = C \det(I + (H - H^0)(H^0 - \lambda)^{-1}), \quad C = \prod_{j=0}^p \frac{a_j^0}{a_j},$$

which is meromorphic on Λ , see [F1]. Recall that $T = 1/\alpha$ is the transmission coefficient in the S -matrix for the pair H, H^0 (see Section 3.1). If α has some poles, then they coincide with some λ_k^\pm . It is well known that if $\alpha(\lambda) = 0$ for some zero $\lambda \in \Lambda_1$, then λ is an eigenvalue of H and $\lambda \in \cup \gamma_k^+$. Note that there are no eigenvalues on the spectrum $\sigma_{ac}(H^0) \subset \Lambda_1$ since $|\alpha(\lambda)| \geq 1$ on $\sigma_{ac}(H^0)$.

We define the functions A, J by

$$J(\lambda) = 2\Omega(\lambda + i0) \operatorname{Im} \alpha(\lambda + i0), \quad A(\lambda) = \operatorname{Re} \alpha(\lambda + i0) - 1, \quad \text{for } \lambda \in \sigma(H^0) \subset \Lambda_1.$$

These functions were introduced for the Schrödinger operator on \mathbb{R} with periodic plus compactly supported potentials by the second author in [K1]. We show that A, J are polynomials on \mathbb{C} and they are real on the real line. Instead of the function α we consider the modified function $\hat{w} = 2i\Omega\alpha$ on Λ . We show that \hat{w} satisfies

$$\hat{w} = 2i\Omega\alpha = 2i\Omega(1 + A) - J \quad \text{on } \Lambda. \quad (1.18)$$

Recall that Ω is analytic on Λ and $\Omega = 0$ for some $\lambda \in \Lambda$ iff $\lambda = \lambda_k^-$ or $\lambda = \lambda_k^+$ for some $k \geq 0$. Then the function \hat{w} is analytic on Λ and has branch points λ_n^\pm if $\gamma_n \neq \emptyset$. The zeros of \hat{w} are the eigenvalues and the resonances. Define the set

$$\Lambda_0 = \{\lambda \in \Lambda : \lambda = \lambda_k^+ \in \Lambda_1 \text{ and } \lambda = \lambda_k^+ \in \Lambda_2, \gamma_k = \emptyset\} \subset \Lambda.$$

In fact with each $\gamma_k = \emptyset$ we associate two points $\lambda_k^+ \in \Lambda_1$ and $\lambda_k^+ \in \Lambda_2$ from the set Λ_0 . If each gap of H^0 is not empty, then $\Lambda_0 = \emptyset$.

Definition 1. *Each zero of \hat{w} on $\Lambda \setminus \Lambda_0$ is a state of H .*

- 1) *A state $\lambda \in \Lambda_1$ is a bound state.*
 - 2) *A state $\lambda \in \Lambda_2$ is a resonance.*
 - 3) *A state $\lambda = \lambda_k^\pm$, $k = 1, \dots, q$ is a virtual state.*
- A resonance $\lambda \in \cup \gamma_k^- \subset \Lambda_2$ is an anti-bound state.*

It is known that the gaps $\gamma_k = \emptyset$ do not give contribution to the states. Recall that S -matrix for H, H^0 is meromorphic on Λ , but it is analytic at the points from Λ_0 (see [F1]). Roughly speaking there is no difference between the points from Λ_0 and other points inside the spectrum of H^0 .

In accordance with the continuous case [K1] we define the important function

$$\mathcal{F}(\lambda) = \hat{w}(\lambda)\hat{w}^*(\lambda), \quad \lambda \in \Lambda_1, \quad (1.19)$$

where we put $f^*(\lambda) := \overline{f(\bar{\lambda})}$. For the perturbation V with $(u, v) \in \mathcal{V}_\nu$ we define the constants

$$c_3 = c_1 c_2, \quad c_1 = \frac{1}{\prod_0^p a_j}, \quad c_2 = \begin{cases} c_1 u_p (a_p^0 + a_p) & \text{if } \nu = 2p, \\ c_1 (a_p^0)^2 v_p & \text{if } \nu = 2p - 1. \end{cases} \quad (1.20)$$

The distribution of the states is summarized in the following theorem (see [IK1]).

Theorem 1.1. *Let the Jacobi operator $H = H^0 + V$ satisfy (1.1)–(1.3). Suppose $(u, v) \in \mathcal{V}_\nu$, where $\nu \in \{2p, 2p-1\}$. Then \hat{w} satisfies (1.18) and the following facts hold true.*

i) *The function $\mathcal{F}(\lambda) = \hat{w}(\lambda)\hat{w}^*(\lambda)$, $\lambda \in \Lambda_1$, is a real polynomial. Each zero of \mathcal{F} is the projection of a state of H on the first sheet. There are no other zeros. The multiplicity of a bound state and a resonance is the multiplicity of its projection as a zero of \mathcal{F} . All bound states are simple. The virtual state at λ_j^\pm , $\gamma_j \neq \emptyset$, $j = 1, \dots, q-1$, is a simple zero of \mathcal{F} . Moreover, \mathcal{F} satisfies*

$$\mathcal{F}(\lambda) = -\lambda^\kappa (c_3 v_0 + \mathcal{O}(\lambda^{-1})), \quad \kappa = \nu + 2q - 1, \quad \lambda \rightarrow \infty, \quad (1.21)$$

here κ is the total number of states (counted with multiplicities).

ii) *There exists an even number of states (counted with multiplicities) on each set $\gamma_j^c \neq \emptyset$, $j = 1, \dots, q-1$, where γ_j^c is a union of the physical gap $\overline{\gamma_j^+}$ and non-physical gap $\overline{\gamma_j^-}$ (see (1.13)).*

iii) *Let $\lambda_1 \in \gamma_j^+$ be a bound state for some $j = 0, \dots, q$, i.e. $\hat{w}(\lambda_1) = 0$. Let $\lambda_2 \in \gamma_j^- \subset \Lambda_2$ be the same number but on the second sheet Λ_2 . Then $\lambda_2 \in \gamma_j^-$ is not an antibound state, i.e. $\hat{w}(\lambda_2) \neq 0$.*

Let f_n^+ denote the Jost solution for the equation $Hy = \lambda y$ satisfying $f_n^+ = \psi_n^+$ for $n \geq p+1$ (see (2.29), (2.30)) and f_0^+ is called the Jost function. We prove that the operator H is uniquely determined by the pair (\hat{w}, f_0^+) . In the following definition we describe the class of functions with characteristic properties of (\hat{w}, f_0^+) .

Definition 2. *For $\nu \in \mathbb{N}$, let \mathfrak{C}_ν denote the class of pairs of functions (w, f) on Λ : w is entire function of the form*

$$w = 2i\Omega(1 + A) - J, \\ w = \begin{cases} \frac{A_p}{c_1} \lambda^q (1 + \mathcal{O}(\lambda^{-1})) & \text{if } \lambda \in \Lambda_1 \\ -\frac{v_0}{A_p} c_2 \lambda^{\nu+q-1} (1 + \mathcal{O}(\lambda^{-1})) & \text{if } \lambda \in \Lambda_2 \end{cases} \quad \text{as } \lambda \rightarrow \infty,$$

where A, J are real polynomials (with real coefficients) of the orders $\nu-1$ and $\nu+q-1$ respectively;

f is meromorphic function of the form

$$f = P_1 + \frac{\phi}{\varphi_q} P_2 + i \frac{\Omega(\lambda)}{\varphi_q} P_2, \\ f(\lambda) = \begin{cases} c_1 A_p + \mathcal{O}(\lambda^{-1}) & \text{if } \lambda \in \Lambda_1 \\ -\frac{c_2}{A_p} \lambda^\nu + \mathcal{O}(\lambda^{\nu-1}) & \text{if } \lambda \in \Lambda_2 \end{cases} \quad \text{as } \lambda \rightarrow \infty,$$

where P_1 and P_2 are real polynomials of the orders $\nu-2$ and $\nu-1$ respectively.

Here A_p is given by

$$A_p = \prod_{n=0}^p a_n^0. \quad (1.22)$$

The real constants c_1, c_2, v_0 satisfy: $c_1 > 0, c_2 \neq 0, v_0 \neq 0$.

We denote $\sigma_{\text{st}} \subset \Lambda$ the set of all zeros of w . Denote $\sigma_{\text{bs}} = \sigma_{\text{st}} \cap \Lambda_1$, $\sigma_{\text{st}} = \{\rho_k\}_{k=1}^N$, and let

$$\mathcal{F} = ww^* = 4(1 - \Delta^2)(1 + A)^2 + J^2, \quad s = \frac{2i\Omega}{a_0^0} f - w.$$

Suppose that the following properties are satisfied:

- 1) $\sigma_{\text{bs}} \in \cup_0^q \gamma_j^+$
- 2) The function \mathcal{F} has even number of zeros on each interval $[\lambda_j^-, \lambda_j^+]$, $j = 1, \dots, q-1$, and \mathcal{F} has only simple zeros at λ_j^\pm , $\gamma_j \neq \emptyset$, and at $\rho_k \in \sigma_{\text{bs}}$, $k = 1, \dots, N$.
- 3) For $k = 1, \dots, N$,

$$g_k = -\frac{D^-}{D^+} \frac{2i\Omega(\rho_k)}{sw'(\rho_k)} > 0. \quad (1.23)$$

- 4) If $\varphi_q(\lambda_j^\pm) = 0$, for some $j = 0, \dots, q$, then $s(\lambda_j^\pm) = w(\lambda_j^\pm)$.

If $(u, v) \in \mathcal{V}_\nu$, then Lemma 3.3 states that $(\hat{w}, f_0^+) \in \mathfrak{E}_\nu$ with $f = f_0^+$, $P_1 = \vartheta_0^+$, $P_2 = \varphi_0^+$, $w = \hat{w}$, $A = A$ and $J = J$. Moreover, $s = \hat{s} := \varphi_q s / a_0^0$ with s defined in (2.36) and $g_k = \gamma_{+,k}$ is the norming constant defined in (3.5).

Now we construct the mapping $\mathfrak{F} : \mathcal{V}_\nu \rightarrow \mathfrak{E}_\nu$, $\nu \in \{2p-1, 2p\}$, by the rule:

$$(u, v) \rightarrow (\hat{w}, f_0^+), \quad (1.24)$$

i.e. to each $(u, v) \in \mathcal{V}_\nu$ we associate $(\hat{w}, f_0^+) \in \mathfrak{E}_\nu$.

Our main result is formulated in the following theorem.

Theorem 1.2. *Let $\kappa = \nu + 2q - 1$, $\nu \in \{2p-1, 2p\}$, and suppose $\kappa \geq 2q + 1$. Then the mapping $\mathfrak{F} : \mathcal{V}_\nu \mapsto \mathfrak{E}_\nu$ given by $\mathfrak{F}(V) = (\hat{w}, f_0^+)$ is one-to-one and onto. Moreover, the reconstruction algorithm is specified.*

In Theorem (1.2) we solve the inverse problem for the mapping \mathfrak{F} . The solution is divided into the following three parts.

1. Uniqueness. Do the pair of functions $(\hat{w}, f_0^+) \in \mathfrak{E}_\nu$ determine uniquely $(u, v) \in \mathcal{V}_\nu$?
2. Reconstruction. Give an algorithm for recovering (u, v) from $(\hat{w}, f_0^+) \in \mathfrak{E}_\nu$ only.
3. Characterization. Give necessary and sufficient conditions for (w, f) to be the Jost function and \hat{w} for some perturbation $(u, v) \in \mathcal{V}_\nu$.

Using the polynomial interpolation as in [IK3], Theorems 1.4 and 1.5, we get that operator H can be reconstructed from H^0 , $\sigma_{\text{st}}(H)$, $\text{Zeros}(R_- + 1)$ and two polynomials A, φ_0^+ which follows from the following theorem.

Theorem 1.3. *Suppose that the functions \hat{w} and $R_- + 1$ have only simple zeros, disjoint from the end-points λ_k^\pm , $k = 0, \dots, q$, and from the points μ_j , $j = 1, \dots, q-1$, such that $\varphi_q(\tilde{\mu}_j) = 0$ (the Dirichlet eigenvalues). Then the pair of functions (\hat{w}, f_0^+) is uniquely determined by the bound states and resonances of H , the set of zeros of the function $R_- + 1$, the polynomials A, φ_0^+ , and the constants c_3, v_0 (see (1.20)).*

Note that from the assumptions of Theorem 1.3 it follows that the set of zeros of \hat{w} coincide with $\sigma_{\text{st}}(H) = \sigma_{\text{bs}}(H) \cup \sigma_{\text{r}}(H)$ (no virtual states are present) and coincide with the zeros of $\alpha = 1/T$. Moreover, the zeros of $R_- + 1$ coincide with the zeros of the Jost function f_0^+ (see the proof Theorem 1.3).

Plan of the paper. In Part 2 we recall the construction of the quasi-momentum map and the associated Riemann surface (Section 2.1), and consider the scattering problem by finitely supported perturbations (Section 2.2).

In Part 3 we prove the Theorems 1.2 and 1.3. In Section 3.1 we summarize the characteristic properties of the scattering data. In Section 3.2 we summarize the inverse scattering results from [Kh1] and [EMT] in the form suitable for us. Finally in Sections 3.3 and 3.4 we prove Theorems 1.2 and 1.3.

2 Preliminaries.

2.1 Quasi-momentum map and Riemann surface \mathcal{Z} .

In this section we recall the construction of the conformal mapping of the Riemann surface onto the plan with “radial slits” \mathcal{Z} , given in [IK1]. Our definition corrects the similar construction in [BE] and [EMT], where there was a mistake.

We suppose that all gaps are open: $\lambda_j^- < \lambda_j^+$, $j = 1, \dots, q-1$.

Introduce a domain $\mathbb{C} \setminus \cup_0^q \overline{\gamma}_j$ and a quasi-momentum domain \mathbb{K} by

$$\mathbb{K} = \{\kappa \in \mathbb{C} : -\pi \leq \text{Re } \kappa \leq 0\} \setminus \cup_1^{q-1} \overline{\Gamma}_j, \quad \Gamma_j = \left(-\frac{\pi j + i h_j}{q}, -\frac{\pi j - i h_j}{q} \right).$$

Here $h_j \geq 0$ is defined by the equation $\cosh h_j = (-1)^{j-q} \Delta(\alpha_j)$ and α_j is a zero of $\Delta'(\lambda)$ in the “gap” $[\lambda_j^-, \lambda_j^+]$. For each periodic Jacobi operator there exists a unique conformal mapping $\kappa : \mathbb{C} \setminus \cup_0^q \overline{\gamma}_j \rightarrow \mathbb{K}$ such that the following identities and asymptotics hold true:

$$\cos q\kappa(\lambda) = \Delta(\lambda), \quad \lambda \in \mathbb{C} \setminus \cup_0^q \overline{\gamma}_j, \quad \text{and} \quad \kappa(it) \rightarrow \pm i\infty \quad \text{as } t \rightarrow \pm\infty. \quad (2.25)$$

The quasi-momentum κ maps the half plane $\mathbb{C}_\pm = \{\lambda \in \mathbb{C} ; \pm \text{Im } \lambda > 0\}$ onto the half-strip $\mathbb{K}_\pm = \mathbb{K} \cap \mathbb{C}_\pm$ and $\sigma_{\text{ac}}(J^0) = \{\lambda \in \mathbb{R} ; \text{Im } \kappa(\lambda) = 0\}$.

Define the two strips \mathbb{K}_S and \mathcal{K} by

$$\mathbb{K}_S = -\mathbb{K} \quad \text{and} \quad \mathcal{K} = \mathbb{K}_S \cup \mathbb{K} \subset \{\kappa \in \mathbb{C} : \text{Re } \kappa \in [-\pi, \pi]\}.$$

The function κ has an analytic continuation from $\Lambda_1 \cap \mathbb{C}_+$ into $\Lambda_1 \cap \mathbb{C}_-$ through the infinite gaps $\gamma_q = (\lambda_q^-, \infty)$ by the symmetry and satisfies:

- 1) κ is a conformal mapping $\kappa : \Lambda_1 \rightarrow \mathcal{K}_+ = \mathcal{K} \cap \mathbb{C}_+$, where we identify the boundaries $\{\kappa = \pi + it, t > 0\}$ and $\{\kappa = -\pi + it, t > 0\}$.
- 2) $\kappa : \Lambda_2 \rightarrow \mathcal{K}_- = \mathcal{K} \cap \mathbb{C}_-$ is a conformal mapping, where we identify the boundaries $\{\kappa = \pi - it, t > 0\}$ and $\{\kappa = -\pi - it, t > 0\}$.
- 3) Thus $\kappa : \Lambda \rightarrow \mathcal{K}$ is a conformal mapping.

Consider the function $z = e^{i\kappa(\lambda)}$, $\lambda \in \Lambda$. The function $z(\lambda)$, $\lambda \in \Lambda$, is a conformal mapping $z : \Lambda \rightarrow \mathcal{Z} = \mathbb{C} \setminus \cup \bar{g}_j$, where the radial cut g_j is given by

$$g_j = (e^{-\frac{h_j}{q} + i\frac{\pi j}{q}}, e^{\frac{h_j}{q} + i\frac{\pi j}{q}}), \quad j = \pm 1, \dots, \pm(q-1).$$

The function $z(\lambda)$, $\lambda \in \Lambda$, maps the first sheet Λ_1 into the “disk” $\mathcal{Z}_1 = \mathcal{Z} \cap \mathbb{D}_1$, $\mathbb{D}_1 = \{z \in \mathbb{C} : |z| < 1\}$, and $z(\cdot)$ maps the second sheet Λ_2 into the domain $\mathcal{Z}_2 = \mathcal{Z} \setminus \mathbb{D}_1$. In fact, we obtain the parametrization of the two-sheeted Riemann surface Λ by the “plane” \mathcal{Z} . Thus below we call \mathcal{Z}_1 also the “physical sheet” and \mathcal{Z}_2 also the “non-physical sheet”.

Note that if all $a_n^0 = 1$, $b_n^0 = 0$, then we have $\lambda = \frac{1}{2}(z + \frac{1}{z})$. This function $\lambda(z)$ is a conformal mapping from the disk \mathbb{D}_1 onto the cut domain $\mathbb{C} \setminus [-2, 2]$.

Now, the functions $\psi^\pm(\lambda)$ can be considered as functions of $z \in \mathcal{Z}$. The functions $\psi_n^\pm(z) \equiv \psi_n^\pm(\lambda(z))$ are meromorphic in \mathcal{Z} with the only possible singularities at the images of the Dirichlet eigenvalues $z(\mu_j) \in \mathcal{Z}$ and at 0. More precisely,

- 1) ψ_n^\pm are analytic in $\mathcal{Z} \setminus (\{z(\mu_j)\}_{j=1}^{q-1} \cup \{0\})$ and continuous up to $\partial\mathcal{Z} \setminus \{z(\mu_j)\}_{j=1}^{q-1}$.
- 2) $\psi_n^\pm(z)$ has a simple pole at $z(\mu_j) \in \mathcal{Z}$ if μ_j is a pole of m_\pm , no pole if μ_j is not a singularity of m_\pm (not a square root singularity if μ_j coincides with the band edge) and if μ_j coincides with the band edge: $\mu_j = \lambda_j^\sigma$, $\sigma = +$ or $\sigma = -$, $j = 1, \dots, q-1$, then

$$\psi_n^\pm(z) = \pm\sigma(-1)^{q-j} \frac{iC(n)}{z - z(\lambda_j^\sigma)} + \mathcal{O}(1), \quad \lambda \in [\lambda_{j-1}^+, \lambda_j^-], \quad (2.26)$$

for some constant $C(n) \in \mathbb{R}$. Note that the sign comes from the analytic continuation of the square root $\Omega(\lambda)$ using the definition (1.14).

3) The following identities hold true:

$$\psi_n^\pm(\bar{z}) = \psi_n^\pm(z^{-1}) = \psi_n^\mp(z) = \overline{\psi_n^\pm(z)} \text{ as } |z| = 1. \quad (2.27)$$

4) The following asymptotics hold true:

$$\psi_n^\pm(z) = (-1)^n \left(\prod_{j=0}^{n-1} {}^*a_j \right)^{\pm 1} z^{\pm n} \left(1 + \mathcal{O}(z) \right) \quad \text{as } z \rightarrow 0.$$

We collect below some properties of the quasi-momentum κ on the gaps.

On each γ_j^+ , $j = 0, 1, \dots, q$, the quasi-momentum $\kappa(\lambda)$ has constant real part and positive $\text{Im } \kappa$:

$$\text{Re } \kappa|_{\gamma_j^+} = -\frac{q-j}{q}\pi, \quad \kappa(\lambda_j^-) = \kappa(\lambda_j^+) = -\frac{q-j}{q}\pi, \quad \text{Im } \kappa|_{\gamma_j^+} > 0.$$

Moreover, as λ increases from λ_j^- to α_j the imaginary part $\text{Im } \kappa \equiv h(\lambda)$ is monotonically increasing from 0 to h_j and as λ increases from α_j to λ_j^+ the imaginary part $\text{Im } \kappa \equiv h(\lambda + i0)$ is monotonically decreasing from h_j to 0. Then

$$\frac{1}{2}\varphi_q(\lambda)(m_+(\lambda) - m_-(\lambda)) = \sqrt{\Delta^2(\lambda) - 1} = i \sin q\kappa(\lambda) = -(-1)^{q-k} \sinh qh(\lambda + i0), \quad (2.28)$$

where $\sinh qh = -2^{-1}(z^q - z^{-q}) > 0$.

2.2 Scattering by finitely supported perturbations.

For finitely-supported perturbation $(u, v) \in \mathcal{V}_\nu$ we define the Jost solutions $f^\pm = (f_n^\pm)_{n \in \mathbb{Z}}$ for the equation

$$a_{n-1}y_{n-1} + a_n y_{n+1} + b_n y_n = \lambda y_n, \quad (\lambda, n) \in \mathbb{C} \times \mathbb{Z}, \quad (2.29)$$

(here $a_n = a_n^0 + u_n, b_n = b_n^0 + v_n$ and $u_n = 0, v_n = 0$ for all $n \notin [0, p]$) by the conditions:

$$f_n^- = \psi_n^-, \text{ for } n \leq 0 \quad \text{and} \quad f_n^+ = \psi_n^+, \text{ for } n \geq p+1. \quad (2.30)$$

Let $\vartheta^\pm, \varphi^\pm$ be solutions to the equation (2.29) satisfying the conditions:

$$\vartheta_n^- = \vartheta_n, \quad \varphi_n^- = \varphi_n \text{ for } n \leq 0 \quad \text{and} \quad \vartheta_n^+ = \vartheta_n, \quad \varphi_n^+ = \varphi_n \text{ for } n \geq p+1. \quad (2.31)$$

Note that each of $\vartheta_n^\pm, \varphi_n^\pm, n \in \mathbb{Z}$ is a polynomial in λ .

The Jost solutions f^\pm inherit the properties of ψ^\pm . We state this properties on the Riemann surface \mathcal{Z} as defined in Sections 2.1.

Lemma 2.1. 1) Each $f_n^\pm, n \in \mathbb{Z}$, is analytic in $\mathcal{Z} \setminus \{0\}$ and continuous up to $\partial\mathcal{Z} \setminus \{z(\mu_j)\}_{j=1}^{q-1}$. Moreover, the following identities hold true:

$$f^\sigma = \vartheta^\sigma + m_\sigma \varphi^\sigma, \quad \sigma = \pm. \quad (2.32)$$

$$f_n^\pm(\bar{z}) = f_n^\pm(z^{-1}) = \overline{f_n^\pm(z)} \quad \text{for} \quad |z| = 1. \quad (2.33)$$

2) $f_n^\pm(z)$ does not have a singularity at $z(\mu_j)$ if μ_j is not a singularity (square root singularity if μ_j coincides with the band edge) of m_\pm , otherwise, $f_n^\pm(z)$ can have either a simple pole at $z(\mu_j)$ if μ_j is a pole of m_\pm , or a square root singularity,

$$f_n^\pm(\lambda) = \pm \sigma (-1)^{q-j} \frac{iC(n)}{\sqrt{\lambda - \lambda_j^\sigma}} + \mathcal{O}(1), \quad \lambda \in [\lambda_{j-1}^+, \lambda_j^-], \quad (2.34)$$

if μ_j coincides with the band edge: $\mu_j = \lambda_j^\sigma$, $\sigma = +$ or $\sigma = -$, $j = 1, \dots, q-1$. Here $C(n)$ is bounded and real, the factor $\sigma(-1)^{q-j}$ comes from the analytic continuation of the square root $\Omega(\lambda)$ using Definition (1.14).

Define the unperturbed Wronskian $\{\cdot, \cdot\}^0$ and the perturbed Wronskian $\{\cdot, \cdot\}$ for sequences $f = (f_n)_{n \in \mathbb{Z}}, g = (g_n)_{n \in \mathbb{Z}}$ by

$$\{f, g\}_n^0 = a_n^0(f_n g_{n+1} - f_{n+1} g_n), \quad \{f, g\}_n = a_n(f_n g_{n+1} - f_{n+1} g_n).$$

Note that if f, g are solutions of (2.29), then the Wronskian $\{f, g\}_n$ is independent of n .

The Jost solutions $\overline{f_n^\pm(\lambda)}$ and $f_n^\pm(\lambda)$, $\lambda \in \text{int } \sigma_{\text{ac}}(H^0)$, are solutions of the same equation $Hf = \lambda f$ and using $\overline{\psi_n^\pm(\lambda)} = \psi_n^\mp(\lambda)$, $\lambda \in \sigma_{\text{ac}}(H^0)$, we have

$$\{f^\pm, \overline{f^\pm}\} = \{\psi^\pm, \psi^\mp\}^0 = a_0^0(m_\mp - m_\pm) = \mp a_0^0 \frac{(z^q - z^{-q})}{\varphi_q}, \quad 2i\Omega(\lambda) = z^q - z^{-q}. \quad (2.35)$$

We denote

$$s = \{f^+, \overline{f^-}\}, \quad w = \{f^-, f^+\}. \quad (2.36)$$

Moreover we have (see [IK1])

$$w = \{f_n^-, f_n^+\} = \text{const} = \{f_0^-, f_0^+\} = a_0(f_0^- f_1^+ - f_1^- f_0^+) = a_0 f_1^+ + (v_0 - a_0^0 m_-) f_0^+, \quad (2.37)$$

$$s = \{f_n^+, \overline{f_n^-}\} = \text{const} = \{f_0^+, \overline{f_0^-}\} = a_0(f_0^+ \overline{f_1^-} - f_1^+ \overline{f_0^-}) = (a_0^0 \overline{m_-} - v_0) f_0^+ - a_0 f_1^+, \quad (2.38)$$

where we have used that $f_0^- = \psi_0^- = 1$ and $a_0 f_1^- = a_0^0 m_- - v_0$, $\overline{m^\pm} = m^\mp$, for $\lambda \in \sigma_{\text{ac}}(H^0)$, by applying the Jacobi equation (1.6).

The following identities hold true:

$$f_n^\pm = \alpha \overline{f_n^\mp} + \beta_\mp f_n^\mp, \quad \lambda \in \text{int } \sigma_{\text{ac}}(H^0), \quad (2.39)$$

where

$$\alpha = \frac{\{f^\mp, f^\pm\}}{\{f^\mp, \overline{f^\mp}\}} = \frac{\varphi_q \{f^-, f^+\}}{a_0^0(z^q - z^{-q})} = \frac{\varphi_q w}{a_0^0(z^q - z^{-q})}, \quad (2.40)$$

$$\beta_- = \frac{\{f^+, \overline{f^-}\}}{\{f^-, \overline{f^-}\}} = \frac{\varphi_q s}{a_0^0(z^q - z^{-q})}, \quad \beta_+ = \frac{\{f^-, \overline{f^+}\}}{\{f^+, \overline{f^+}\}} = \frac{\varphi_q \overline{s}}{a_0^0(z^q - z^{-q})} \quad (2.41)$$

since $\overline{s} = -\{f^-, \overline{f^+}\}$. Using (2.27), we get $\overline{\beta}_\pm = -\beta_\mp$ for $\lambda \in \text{int } \sigma_{\text{ac}}(H^0)$ and

$$|\alpha(\lambda)|^2 = 1 + |\beta_\pm(\lambda)|^2, \quad \lambda \in \text{int } \sigma_{\text{ac}}(H^0). \quad (2.42)$$

Applying (2.40), (2.41) in (2.42) we get

$$w(z)w(z^{-1}) + \left(\frac{a_0}{\varphi_q(z)}\right)^2 (z^q - z^{-q})^2 = s(z)s(z^{-1}), \quad |z| = 1, \quad z^2 \neq 1. \quad (2.43)$$

We define the scattering matrix

$$S(\lambda) = \begin{pmatrix} T(\lambda) & R_-(\lambda) \\ R_+(\lambda) & T(\lambda) \end{pmatrix}, \quad \lambda \in \sigma(H^0), \quad (2.44)$$

for the pair (H, H^0) , where

$$T(\lambda) = \frac{1}{\alpha(\lambda)}, \quad R_\pm(\lambda) = \frac{\beta_\pm(\lambda)}{\alpha(\lambda)} = \frac{\mp \{f^\mp(\lambda), \overline{f^\pm(\lambda)}\}}{\{f^-(\lambda), f^+(\lambda)\}}.$$

We have also

$$R_+ = \frac{\overline{s}}{w}, \quad R_- = \frac{s}{w}.$$

The matrix $S(\lambda)$ is unitary: $|T(\lambda)|^2 + |R_\pm(\lambda)|^2 = 1$, $T(\lambda)\overline{R_+(\lambda)} = -\overline{T(\lambda)}R_-(\lambda)$, and extends to Λ as a meromorphic function. The quantities T and R_\pm are the transmission and the reflection coefficients respectively:

$$T(\lambda)f_n^\pm(\lambda) = \begin{cases} T(\lambda)\psi_n^\pm(\lambda), & n \rightarrow \pm\infty \\ \psi_n^\pm(\lambda) + R_\mp(\lambda)\psi_n^\mp(\lambda), & n \rightarrow \mp\infty \end{cases}, \quad \lambda \in \sigma(H^0).$$

The determinant of the scattering matrix is given by

$$\det S(\lambda) = T^2 - R_+ R_- = \frac{1}{\alpha^2} + \frac{|\beta|^2}{\alpha^2} = \frac{|\alpha|^2}{\alpha^2} = \frac{\overline{\alpha(\lambda)}}{\alpha(\lambda)}.$$

3 Inverse resonance problem.

In this section we prove the Theorems 1.2 and 1.3.

We consider the scattering data (\hat{w}, f_0^+) . In Section 3.1 we summarize the characteristic properties of the scattering data and show (see Lemma 3.3) that, if $V \equiv (u, v) \in \mathcal{V}_\nu$, then $(\hat{w}, f_0^+) \in \mathfrak{C}_\nu$.

We will show how from this data we can reconstruct the reflection coefficients and the norming constants which define a unique Jacobi operator H .

In Section 3.2 we give an account of the inverse scattering results from [Kh1] and [EMT] in the form suitable for us.

In Section 3.3 we prove Theorem 1.2. In Section 3.4 we give a sketch of the proof of Theorem 1.3.

3.1 Characteristic properties of the scattering data.

In this section we summarize the properties of the scattering data (\hat{w}, f_0^+) which are needed for the proof of our main results.

Let $M_\pm \in \mathbb{C}$ denote (the projection of) the set of poles of m_\pm . Let M_e denote the set of square root singularities of m_\pm if $\mu_k = \lambda_j^\pm$, $j = 1, \dots, q-1$. Note that $M_+ \cap M_- = \emptyset$. We put

$$\varphi_q = a_0^0 D^+ D^-, \quad D^+ = \prod_{\mu_k \in M_+ \cup M_e} (\tilde{\lambda} - \mu_k), \quad D^- = \prod_{\mu_k \in M_-} (\tilde{\lambda} - \mu_k),$$

where $\tilde{\cdot} : \Lambda \mapsto \mathbb{C}$ is the natural projection introduced in (1.12). Note that this definition of D^\pm differs from that used in [IK1]. We mark with $\hat{\cdot}$ the modified (regularized) quantities: $\hat{\psi}^\pm = D^\pm \psi^\pm$, $\hat{f}^\pm = D^\pm f^\pm$, $\hat{w} = \frac{\varphi_q}{a_0^0} w$, which are analytic in Λ_1 . We denote also $\hat{s} = \frac{\varphi_q}{a_0^0} s$, the meromorphic function on Λ_1 . Note that the function $(D^+)^2 s$ is analytic on Λ_1 .

In [IK1] we proved the following result.

Lemma 3.1. *Let $2i\Omega(\lambda) = 2i \sin q\kappa(\lambda) = z^q - z^{-q}$ and $\lambda \in \sigma_{ac}(H^0)$. The following identities*

hold true.

$$\begin{aligned}
w &= a_0 f_1^+ + (v_0 - a_0^0 m_-) f_0^+, \quad s = (a_0^0 m_+ - v_0) f_0^+ - a_0 f_1^+, \quad \hat{w} = \frac{\varphi_q}{a_0^0} w, \quad \hat{s} = \frac{\varphi_q}{a_0^0} s, \\
\hat{w} \hat{w}^* &= 4\Omega^2 + \hat{s} \hat{s}^*, \quad w + s = \frac{2i\Omega}{\varphi_q} f_0^+, \quad \hat{w} + \hat{s} = \frac{2i\Omega}{a_0^0} f_0^+, \\
\hat{w} &= 2i\Omega(1 + A) - J, \quad \hat{s} = -2i\Omega(1 + \tilde{A}) + \tilde{J}, \\
A &= \frac{1}{2} \left(\frac{a_0}{a_0^0} \varphi_1^+ + \frac{v_0}{a_0^0} \varphi_0^+ + \vartheta_0^+ \right) - 1 = \frac{1}{2} \left[\left(\frac{a_0}{a_0^0} \varphi_1^+ - \varphi_1 \right) + \frac{v_0}{a_0^0} \varphi_0^+ + (\vartheta_0^+ - \vartheta_0) \right], \\
\tilde{A} &= \frac{1}{2} \left(\frac{a_0}{a_0^0} \varphi_1^+ + \frac{v_0}{a_0^0} \varphi_0^+ - \vartheta_0^+ - \frac{2\phi}{\varphi_q} \varphi_0^+ \right) - 1, \\
J &= - \left[\frac{a_0}{a_0^0} \varphi_q \vartheta_1^+ + \phi \left(\frac{a_0}{a_0^0} \varphi_1^+ - \vartheta_0^+ \right) + \frac{v_0}{a_0^0} (\varphi_q \vartheta_0^+ + \phi \varphi_0^+) + \vartheta_{q+1} \varphi_0^+ \right], \quad \vartheta_{q+1} \varphi_0^+ = - \frac{\phi^2 + \Omega^2}{\varphi_q} \varphi_0^+, \\
\tilde{J} &= - \left[\frac{a_0}{a_0^0} \varphi_q \vartheta_1^+ + \phi \left(\frac{a_0}{a_0^0} \varphi_1^+ - \vartheta_0^+ \right) + \frac{v_0}{a_0^0} (\varphi_q \vartheta_0^+ + \phi \varphi_0^+) - \frac{\phi^2 - \Omega^2}{\varphi_q} \varphi_0^+ \right], \\
A - \tilde{A} &= \vartheta_0^+ + \frac{\phi}{\varphi_q} \varphi_0^+, \quad J - \tilde{J} = \frac{2\Omega^2}{\varphi_q} \varphi_0^+, \\
4\Omega^2(1 + A)^2 + J^2 &= 4\Omega^2 + 4\Omega^2(1 + \tilde{A})^2 + \tilde{J}^2.
\end{aligned} \tag{3.1}$$

Remark. From Lemma 3.1 it follows that A, J are polynomials, whereas functions \tilde{A}, \tilde{J} are rational with simple poles at $\mu_n \neq \lambda_j^\pm$, $n = 1, \dots, q-1$, $j = 1, \dots, q$, where μ_n is a zero of φ_q and λ_j^\pm is an endpoint of a finite gap.

Note also that φ_j, ϑ_j , $j = 1, 2$ are related via the wronskian property: $a_0(\vartheta_0^+ \varphi_1^+ - \vartheta_1^+ \varphi_0^+) = a_0^0(\vartheta_0 \varphi_1 - \vartheta_1 \varphi_0) = a_0^0$.

In order to define the class of scattering data we need the properties summarized in the following lemma.

Lemma 3.2. Suppose $(u, v) \in \mathcal{V}_\nu$, where $\nu \in \{2p, 2p-1\}$ and $\{\rho_k\}_{k=1}^N = \sigma_{\text{bs}}(H)$ is the set of bound states of H . Let $\{\mu_j\}_{j=1}^{q-1}$ be the set of zeros of φ_q . Then for all $k = 1, \dots, N$ the functions \hat{f}^\pm , \hat{w} and \hat{s} satisfy the following properties:

$$1) \quad \hat{f}_n^\pm(\rho_k) = \beta_\mp(\rho_k) \frac{D^\pm(\rho_k)}{D^\mp(\rho_k)} \hat{f}_n^\mp(\rho_k), \quad \hat{s}(\rho_k) \overline{\hat{s}(\rho_k)} = -4(1 - \Delta^2(\rho_k)). \tag{3.3}$$

$$2) \quad \text{Denote } c_k^\pm = \frac{\hat{f}^\pm(\rho_k)}{\hat{f}^\mp(\rho_k)}, \quad \gamma_{\pm, k} = \left(\sum_{n \in \mathbb{Z}} |\hat{f}_n^\pm(\rho_k)|^2 \right)^{-1}. \quad \text{Then}$$

$$c_k^\pm = \beta_\mp \frac{D^\pm}{D^\mp}(\rho_k), \quad \gamma_{\pm, k} = - (c_k^\pm \hat{w}'(\rho_k))^{-1} > 0, \quad \gamma_{+, k} \gamma_{-, k} = (\hat{w}'(\rho_k))^{-2}. \tag{3.4}$$

3) For $k = 1, \dots, N$, if $\rho_k \in \gamma_j^+$ for some $j = 0, \dots, q$, then

$$\gamma_{+, k} = - \frac{D^-}{D^+} \frac{2i\Omega(\rho_k)}{\hat{s} \hat{w}'(\rho_k)} = \frac{D^-}{D^+} \frac{2(-1)^{q-j} \sinh qh(\rho_k)}{\hat{s} \hat{w}'(\rho_k)} > 0, \tag{3.5}$$

where $h(\rho_k) = \text{Im } \varkappa(\rho_k) > 0$.

4) If $\mu_j \neq \lambda_j^\pm$ for all $j = 1, \dots, q-1$, then we have

$$\hat{s}(\lambda_j^\pm) = \overline{\hat{s}(\lambda_j^\pm)} = \tilde{J}(\lambda_j^\pm) = -\hat{w}(\lambda_j^\pm) = J(\lambda_j^\pm); \quad (3.6)$$

if $\mu_j = \lambda_j^\sigma$, for some $j = 1, \dots, q-1$ and $\sigma = +$ or $\sigma = -$, then

$$\hat{s}(\lambda_j^\sigma) = \tilde{J}(\lambda_j^\sigma) = \hat{w}(\lambda_j^\sigma) = -J(\lambda_j^\sigma). \quad (3.7)$$

5) The function \hat{w} is real on \mathbb{R} and

$$|\hat{w}(z)| \geq |z^q - z^{-q}| \quad \text{for any } |z| = 1. \quad (3.8)$$

Proof. 1) Relations in (3.3) follow from the analytic continuation of the identities (2.39) and (2.42) (see also (3.2)) as $\alpha(\rho_j) = 0$.

2) Formulas in (3.4) come from (6.11) in [EMT].

3) Formula (3.5) follows using that $\beta_- = \frac{D^+ D^-}{(z^q - z^{-q})} s(z) = \frac{1}{(z^q - z^{-q})} \frac{D^-}{D^+} \hat{s}(z)$ and (2.28).

4) Proof of (3.6).

Suppose $\mu_j \neq \lambda_j^\pm$ for all $j = 1, \dots, q-1$. Then $m_-(\lambda_j^\pm) = \overline{m_-(\lambda_j^\pm)}$ and (3.6) follows from (2.37), (2.38). Moreover, as $f^\pm(\lambda_j^\pm)$ are real, it follows also that $\hat{s}(\lambda_j^\pm) = -\hat{w}(\lambda_j^\pm)$.

Now, suppose $\mu_j = \lambda_j^-$, for some $j = 1, \dots, q-1$. Then $f_n^\pm(\lambda)$ are pure imaginary in the limit $\lambda \rightarrow \lambda_j^-$ (see (2.34)) and we use the formulas $w(\lambda) = \{f_n^-, f_n^+\}$, $s(\lambda) = \{f_n^+, \overline{f_n^-}\}$. Taking the limit $\lambda \rightarrow \lambda_j^-$ we get $\hat{s}(\lambda_j^-) = \hat{w}(\lambda_j^-)$. The case $\mu_j = \lambda_j^+$ follows similarly.

5) The property (3.8) $|\hat{w}(z)| \geq |z^q - z^{-q}|$ for $|z| = 1$ follows from $\bar{z} = z^{-1}$ and $\overline{(z^q - z^{-q})} = -(z^q - z^{-q})$ as in (3.9) we have $(z^q - z^{-q})^2 = -|z^q - z^{-q}|^2$. □

Using Lemma 3.2 and the asymptotics of the Jost functions given in [IK1] we have the following result.

Lemma 3.3. *If $(u, v) \in \mathcal{V}_\nu$, $\nu \in \{2p-1, 2p\}$, then $(\hat{w}, f_0^+) \in \mathfrak{C}_\nu$ and*

$$\begin{aligned} \hat{w} = 2i\Omega(1+A) - J &= \begin{cases} \frac{A_p}{c_1} \lambda^q (1 + \mathcal{O}(\lambda^{-1})) & \text{if } \lambda \in \Lambda_1 \\ -\frac{v_0}{A_p} c_2 \lambda^{\nu+q-1} (1 + \mathcal{O}(\lambda^{-1})) & \text{if } \lambda \in \Lambda_2 \end{cases} \quad \text{as } \lambda \rightarrow \infty, \\ f_0^+ = v_0^+ + \frac{\phi}{\varphi_q} \varphi_0^+ + i \frac{\Omega(\lambda)}{\varphi_q} \varphi_0^+ &= \begin{cases} c_1 A_p + \mathcal{O}(\lambda^{-1}) & \text{if } \lambda \in \Lambda_1 \\ -\frac{c_2}{A_p} \lambda^\nu + \mathcal{O}(\lambda^{\nu-1}) & \text{if } \lambda \in \Lambda_2 \end{cases} \quad \text{as } \lambda \rightarrow \infty, \end{aligned}$$

with the constants c_1, c_2 given in (1.20) and A_p defined in (1.22).

The polynomials $1+A$, J have asymptotics

$$1+A = -\frac{c_2}{2A_p} v_0 \lambda^{\nu-1} (1 + \mathcal{O}(\lambda^{-1})), \quad J = \frac{c_2}{2A_p} v_0 \lambda^{\nu+q-1} (1 + \mathcal{O}(\lambda^{-1})).$$

At last we reformulate some properties of the functions \hat{w} and \hat{s} on the Riemann surface \mathcal{Z} .

Lemma 3.4. *Let $(u, v) \in \mathcal{V}_\nu$, $\nu \in \{2p-1, 2p\}$. Then the function $z^q \hat{w}(z)$, is entire in \mathcal{Z} , the function $z^{q-1} \hat{s}(z)$ is analytic in \mathcal{Z}_1 and have poles in \mathcal{Z}_2 . The functions \hat{w} , \hat{s} satisfy the functional equation*

$$\hat{w}(z)\hat{w}(z^{-1}) + (z^q - z^{-q})^2 = \hat{s}(z)\hat{s}(z^{-1}), \quad |z| = 1, \quad z^2 \neq 1. \quad (3.9)$$

Moreover, the following asymptotics hold

$$\begin{aligned} \hat{w} &= \frac{A_p}{c_1} z^{-q} [1 + \mathcal{O}(z)], & \hat{s} &= -\frac{A_p}{c_1} v_0 z^{1-q} [1 + \mathcal{O}(z)] \quad \text{as } z \rightarrow 0, \\ \hat{w} &= -\frac{c_2}{A_p} v_0 z^{\nu+q-1} [1 + \mathcal{O}(z^{-1})], & \hat{s} &= \frac{c_2}{A_p} z^{\nu+q} [1 + \mathcal{O}(z^{-1})] \quad \text{as } z \rightarrow \infty, \end{aligned}$$

with the constants c_1, c_2 given in (1.20) and A_p is defined in (1.22).

Proof. Identity (3.9) follows from (2.43) using that for $|z| = 1$ we have $\varphi_q(z^{-1}) = \varphi(\bar{z}) = \varphi(z)$. The asymptotics of \hat{w} follows from Lemma 3.3 using $2\Delta = z^q + z^{-q} = \lambda^q + \mathcal{O}(\lambda^{q-1})$ as $\lambda \rightarrow \infty$. Similarly follows the asymptotics for \hat{s} . \square

3.2 Inverse scattering problem

We consider the relation between the **left/right scattering data** $S_\pm(H)$ for H ,

$$S_\pm(H) = \{R_\pm(z), \quad z \in \mathbb{S}^1; \quad \rho_k, \gamma_{\pm,k} > 0, \quad k = 1, \dots, N\},$$

and the perturbation coefficients (u, v) in the Jacobi operator H .

In [Kh1] and [EMT] the inverse scattering problem was solved for Jacobi operators which are short range perturbations of periodic (quasi-periodic in [EMT]) finite-gap operators. Here we give a short summary of their results in the context of periodic background **with finitely supported perturbations**. In this case the proofs follows straightforward from the methods in [EMT] and we omit them.

First we consider the direct problem. Let \mathbb{S}^1 denote the unit circle $|z| = 1$ and consider the measure on \mathbb{S}^1

$$d\omega(z) = \prod_{j=1}^{q-1} \frac{\lambda(z) - \mu_j}{\lambda(z) - \alpha_j} \frac{dz}{z}, \quad (3.10)$$

where $\alpha_j \in \gamma_j$ is the zero of $\Delta'(\lambda)$ (see Section 2.1 and [EMT]).

Introduce the transformation operator K_\pm by

$$(K_\pm h)_n = \sum_{m=n}^{\pm\infty} K_\pm(n, m) h_m, \quad h = (h_n)_{n \in \mathbb{Z}},$$

where the kernel $K_{\pm}(n, m)$ is given by

$$K_{\pm}(n, m) = \frac{1}{2\pi i} \int_{|z|=1} f_n^{\pm}(z) \psi_m^{\mp} d\omega(z). \quad (3.11)$$

The kernels $K_{\pm}(n, m)$, $n, m \in \mathbb{Z}$, are the Fourier coefficients of the Jost solution f_n^{\pm} with respect to the orthonormal system $\{\psi_n^{\pm}\}_{n \in \mathbb{Z}}$ in the Hilbert space $L^2(\mathbb{S}^1, \frac{1}{2\pi i} d\omega)$.

Lemma 3.5. *Assume $(u, v) \in \mathcal{V}_{\nu}$. Then the Jost solutions f^{\pm} have the form*

$$f_n^{\pm} = \sum_{m=n}^{\pm\infty} K_{\pm}(n, m) \psi_m^{\pm}, \quad |z| = 1, \quad (3.12)$$

where the kernels $K_{\pm}(n, m)$ of the finite rank operator K_{\pm} satisfy

$$K_{\pm}(n, m) = 0, \quad \text{for } \pm m < \pm n,$$

and

$$|K_+(n, m)| \leq C \sum_{j=\max\{M, 0\}}^p Q_j, \quad |K_-(n, m)| \leq C \sum_{j=0}^{\min\{M, p\}} Q_j, \quad \text{for } \pm m > \pm n, \quad (3.13)$$

where $Q_j = |u_j| + |v_j|$ and $M = \lceil \frac{n+m}{2} \rceil + 1$, and the constant $C \equiv C(H^0)$ depends on the unperturbed periodic operator H^0 .

Lemma 3.6. *Assume $(u, v) \in \mathcal{V}_{\nu}$. Then $a_n, b_n, n \in \mathbb{Z}$, satisfy*

$$\frac{a_n}{a_n^0} = \frac{K_+(n+1, n+1)}{K_+(n, n)} = \frac{K_-(n, n)}{K_-(n+1, n+1)}, \quad (3.14)$$

$$v_n = a_n^0 \frac{K_+(n, n+1)}{K_+(n, n)} - a_{n-1}^0 \frac{K_+(n-1, n)}{K_+(n-1, n-1)} = a_{n-1}^0 \frac{K_-(n, n-1)}{K_-(n, n)} - a_n^0 \frac{K_-(n+1, n)}{K_-(n+1, n+1)}.$$

Let

$$F^{\pm}(l, m) = F_0^{\pm}(l, m) + \sum_{j=1}^N \gamma_{\pm, j} \hat{\psi}_l^{\pm}(\rho_j) \hat{\psi}_m^{\pm}(\rho_j), \quad (3.15)$$

$$F_0^{\pm}(l, m) = \frac{1}{2\pi i} \int_{|z|=1} R_{\pm}(z) \psi_l^{\pm}(z) \psi_m^{\pm}(z) d\omega(z). \quad (3.16)$$

Note that $F_0^{\pm}(l, m) = F_0^{\pm}(m, l)$ is real. The function $K_{\pm}(n, m)$ satisfies the equation

$$K_{\pm}(n, m) + \sum_{l=n}^{\pm\infty} K_{\pm}(n, l) F_0^{\pm}(l, m) = \frac{\delta(n, m)}{K_{\pm}(n, n)} - \sum_{j=1}^N \gamma_{\pm, j} \hat{f}_n^{\pm}(\rho_j) \hat{\psi}_m^{\pm}(\rho_j). \quad (3.17)$$

We define the Gel'fand-Levitan-Marchenko operator

$$(\mathcal{F}_n^{\pm} f)(j) = \sum_{l=0}^{\infty} F^{\pm}(n \pm l, n \pm j) f_l, \quad f = (f_l)_0^{\infty} \in \ell^{\infty}(0, \infty).$$

Theorem 3.1. *The kernel $K_{\pm}(n, m)$ of the transformation operator satisfies the Gel'fand-Levitan-Marchenko equation*

$$(1 + \mathcal{F}_n^{\pm})K_{\pm}(n, n \pm \cdot) = (K_{\pm}(n, n))^{-1}\delta_0, \quad (3.18)$$

where $K_{\pm}(n, n) = \langle \delta_0, (1 + \mathcal{F}_n^{\pm})^{-1}\delta_0 \rangle^{\frac{1}{2}}$, and we have

$$|F^+(n, m)| \leq C \sum_{j=\lfloor \frac{n+m}{2} \rfloor + 1}^p (|u_j| + |v_j|), \quad |F^-(n, m)| \leq C \sum_{j=0}^{\lfloor \frac{n+m}{2} \rfloor - 1} (|u_j| + |v_j|), \quad (3.19)$$

where the constant $C = C(H^0)$ is of the same nature as in (3.13).

Now we recall the procedure which allows the reconstruction of the perturbation coefficients (u, v) for the Jacobi operator H from the **left/right scattering data** $S_{\pm}(H)$ for H ,

$$S_{\pm}(H) = \{R_{\pm}(z), |z| = 1; \rho_k, \gamma_{\pm, k} > 0, k = 1, \dots, N\}.$$

We summarize the properties of the scattering data $S_{\pm}(H)$ for H with the finitely supported perturbation coefficients $(u, v) \in \mathcal{V}_{\nu}$.

Hypothesis 1 *The scattering data $S_{\pm}(H)$ satisfy the following conditions:*

(i) *The reflection coefficients $R_{\pm}(z)$ are continuous except possibly at $z_l = z(E_l)$, where $\{E_l\}_{l=0}^{2q-1} \equiv \{\lambda_k^{\pm}\}_{k=0}^q$ and fulfill $\overline{R_{\pm}(z)} = R_{\pm}(\overline{z})$. Moreover, $|R_{\pm}(z)| < 1$ for $z \neq z_l$ and*

$$1 - |R_{\pm}(z)|^2 \geq C \prod_{l=0}^{2q-1} |z - z_l|^2.$$

The functions $F_{\pm}(n, m)$ satisfy

$$F_{\pm}(n, m) = 0 \text{ for } \pm(n + m) \geq M. \quad (3.20)$$

for some $M \in \mathbb{Z}$.

(ii) *The values $\rho_k \in \mathbb{R} \setminus \sigma_{ac}(H^0)$, $1 \leq k \leq N$, are distinct and the norming constants $\gamma_{\pm, k}$, $1 \leq k \leq N$, are positive.*

(iii) *$T(z)$ defined via Poisson-Jensen type formula ((6.25) in [EMT]) extends to a single values function on \mathcal{Z}_1 (i.e. it has equal values on the corresponding slits).*

(iv) *Transmission and reflection coefficients satisfy*

$$\lim_{z \rightarrow z_l} \sqrt{\Delta^2(z) - 1} \frac{R_{\pm}(z) + 1}{T(z)} = 0, \quad z_l \neq z(\mu_j), \quad j = 1, \dots, q-1,$$

$$\lim_{z \rightarrow z_l} \sqrt{\Delta^2(z) - 1} \frac{R_{\pm}(z) - 1}{T(z)} = 0, \quad z_l = z(\mu_j), \quad j = 1, \dots, q-1,$$

where $z_l = z(E_l)$, and the consistency conditions

$$\frac{R_-(z)}{R_+(\overline{z})} = -\frac{T(z)}{T(\overline{z})}, \quad \gamma_{+, k} \gamma_{-, k} = \frac{(\text{Res}_{\rho_k} T(\lambda))^2}{4(\Delta^2(\rho_k) - 1)}.$$

Theorem 3.2. *Suppose that Hypothesis 1 is satisfied. Then, for $n \in \mathbb{Z}$, the Gel'fand-Levitan-Marchenko operator $\mathcal{F}_n^\pm : \ell^2 \rightarrow \ell^2$ has finite rang. Moreover, $1 + \mathcal{F}_n^\pm$ is positive and hence invertible.*

In particular, the Gel'fand-Levitan-Marchenko equation (3.18) has a unique solution and $S_+(H)$ or $S_-(H)$ uniquely determines H and the finitely supported perturbation (u, v) .

Inverse problem. If S_\pm (satisfying Hypothesis 1 (i),(ii)) and H^0 are known, we can construct $F^\pm(l, m)$ via formula (3.15) and thus derive the Gel'fand-Levitan-Gel'fand-Levitan-Marchenko equation, which has a unique solution by Theorem 3.2. This solution

$$K_\pm(n, n) = \langle \delta_0, (1 + \mathcal{F}_n^\pm)^{-1} \delta_0 \rangle^{\frac{1}{2}}, \quad K_\pm(n, n \pm j) = \frac{1}{K_\pm(n, n)} \langle \delta_j, (1 + \mathcal{F}_n^\pm)^{-1} \delta_0 \rangle^{\frac{1}{2}}$$

is the kernel of the transformation operator. Since $1 + \mathcal{F}_n^\pm$ is positive, $K_\pm(n, n)$ is positive and we can set in accordance with Lemma 3.6

$$a_n^+ = a_n^0 \frac{K_+(n+1, n+1)}{K_+(n, n)}, \quad b_n^+ = b_n + a_n^0 \frac{K_+(n, n+1)}{K_+(n, n)} - a_{n-1}^0 \frac{K_+(n-1, n)}{K_+(n-1, n-1)}, \quad (3.21)$$

$$a_n^- = a_n^0 \frac{K_-(n, n)}{K_-(n+1, n+1)}, \quad b_n^- = b_n + a_{n-1}^0 \frac{K_-(n, n-1)}{K_-(n, n)} - a_n^0 \frac{K_-(n+1, n)}{K_-(n+1, n+1)}. \quad (3.22)$$

Let H^+ , H^- be the associated Jacobi operators.

Lemma 3.7. *Suppose that a given set S_\pm satisfies Hypothesis 1 (i)-(ii). Then the sequences $(a_n^\pm - a_n^0, b_n^\pm - b_n^0)_{n \in \mathbb{Z}}$, defined in (3.21), (3.22) have finite support. Moreover, $f_n^\pm = \sum_{m=n}^{\pm\infty} K_\pm(n, m) \psi_m^\pm$, where $K_\pm(n, m)$ is the solution of the Gel'fand-Levitan-Marchenko equation (3.18), satisfies $H f^\pm = \lambda f^\pm$ and $f_n^\pm = \psi_n^\pm$ for $\pm n > n^\pm$, for some $n^\pm \in \mathbb{N}$.*

Now in [EMT] it is shown that $a_n^+ = a_n^-$ and $b_n^+ = b_n^-$ and we have

Theorem 3.3. *Hypothesis 1 is necessary and sufficient for a set S_\pm to be left/right scattering data of a unique Jacobi operator H associated with sequences a, b such that $(u, v) \in \mathcal{V}_\nu$.*

We set

$$\frac{a_n^0 + u_n}{a_n^0} = \frac{K_+(n+1, n+1)}{K_+(n, n)}, \quad v_n = a_n^0 \frac{K_+(n, n+1)}{K_+(n, n)} - a_{n-1}^0 \frac{K_+(n-1, n)}{K_+(n-1, n-1)}, \quad (3.23)$$

3.3 Proof of Theorem 1.2.

Let $(u, v) \in \mathcal{V}_\nu$. Then the scattering data satisfy Hypothesis 1. By Lemma 3.2 we have also $(\hat{w}, f_0^+) \in \mathfrak{C}_\nu$. Lemma 3.2 yields the norming constants $\gamma_{\pm, k}$, $k = 1, \dots, N$. The following lemma shows the inverse relation.

Lemma 3.8. Suppose $(\hat{w}, f_0^+) \in \mathfrak{C}_\nu$, $\nu \in \{2p-1, 2p\}$, and (u, v) be defined by (3.23). Then

- 1) Hypothesis 1 is satisfied.
- 2) $(u, v) \in \mathcal{V}_\nu$. Moreover, $f_n^\pm = \sum_{m=n}^{\pm\infty} K_\pm(n, m) \psi_m^\pm$, where $K_\pm(n, m)$ is the solution of the Gel'fand-Levitan-Marchenko equation (3.18), satisfies $Hf^\pm = \lambda f^\pm$ and $f_n^- = \psi_n^-$ for $n \leq 0$, $f_n^+ = \psi_n^+$ for $n \geq p+1$.

Proof.

If $(\hat{w}, f_0^+) \in \mathfrak{C}_\nu$, then we have

$$R_+ = \frac{s(z^{-1})}{w(z)} = \frac{\hat{s}(z^{-1})}{\hat{w}(z)}, \quad |z| = 1.$$

1) We need to check that the scattering data $S_\pm(H)$ satisfies Hypothesis 1.

(i) As $R_- = s/w$, then we have

$$1 - |R_-|^2 = \frac{a_0^0 |z^q - z^{-q}|^2}{\varphi_q^2 |w|^2} \equiv |T|^2 = \frac{|z^q - z^{-q}|^2}{|\hat{w}|^2} \geq C \prod_{l=1}^{2q-1} |z - z(E_l)|^2.$$

It follows from the fact that $\sup \hat{w} < \text{const}$ for $|z| = 1$.

We prove (3.20). If $(u, v) \in \mathcal{V}_\nu$ then for $l+m \geq 2p$ we have

$$F_0^+(l, m) = - \sum_{j=1}^N \gamma_{+,j} \hat{\psi}_l^+(\rho_j) \hat{\psi}_m^+(\rho_j)$$

(see proof of 2) below) and

$$F^+(l, m) = F_0^+(l, m) + \sum_{j=1}^N \gamma_{+,j} \hat{\psi}_l^+(\rho_j) \hat{\psi}_m^+(\rho_j) = 0, \quad l+m \geq 2p,$$

and similar for F^- .

(ii) follows from 3) in the definition of \mathfrak{C}_ν .

(iii) follows as the transmission coefficient T is defined via $T(z) = \frac{z^q - z^{-q}}{\hat{w}(z)}$.

(iv) We have [EMT]

$$\sqrt{\Delta^2(z) - 1} \frac{R_+(z) + 1}{T(z)} = \frac{w(z) + s(z^{-1})}{2} \prod_{j=1}^{q-1} (\lambda - \mu_j)$$

$$\sqrt{\Delta^2(z) - 1} \frac{R_-(z) + 1}{T(z)} = \frac{(w + s)}{2} \prod_{j=1}^{q-1} (\lambda - \mu_j).$$

If $\mu_j \neq E_l$ then f^\pm are continuous and real at $\lambda = E_l$ and the two Wronskians cancel (see [EMT]). This also follows from Lemma 3.2, equation (3.6).

Otherwise, if $\mu_j = E_l$, the Wronskians are purely imaginary (by property (2.34)) and add up. This also follows from Lemma 3.2, equation (3.7).

We have for $|z| = 1$

$$\frac{R_-(z)}{R_+(\bar{z})} = \frac{s(z)w(\bar{z})}{w(z)\overline{s(\bar{z})}}, \quad \frac{T(z)}{T(\bar{z})} = -\frac{\varphi_q(\bar{z})w(\bar{z})}{\varphi_q(z)w(z)},$$

which together with $\overline{s(z)} = s(\bar{z})$, $\overline{w(z)} = s(\bar{w})$ and $\varphi_q(\bar{z}) = \varphi_q(z)$ for $|z| = 1$ give the first consistency condition.

Now we have

$$4(\Delta^2 - 1) = (z^q - z^{-q})^2, \quad T(z) = \frac{z^q - z^{-q}}{\hat{w}(z)} \Rightarrow T^2 = \frac{4(\Delta^2 - 1)}{\hat{w}^2},$$

$$\hat{w}(\lambda) = \hat{w}'(\rho_j)(\lambda - \rho_j) + \mathcal{O}(\lambda - \rho_j)^2 \Rightarrow (\text{Res}_{\rho_j} T)^2 = \frac{4(\Delta^2(\rho_j) - 1)}{(\hat{w}'(\rho_j))^2}.$$

As from Lemma 3.2, equation (3.4), we have

$$\gamma_{+,j}\gamma_{-,j} = (\hat{w}'(\rho_j))^{-2},$$

then we have the second consistency condition.

2) Recall Formula (3.16)

$$F_0^+(l, m) = \frac{1}{2\pi i} \int_{|z|=1} R_+(z)\psi_l^+(z)\psi_m^+(z)d\omega(z),$$

where $d\omega(z)$ is given in (3.10).

Observe that $d\omega$ is meromorphic on \mathcal{Z}_1 with simple pole at $z = 0$. In particular, there are no poles at $z(\alpha_j)$. To evaluate the integral we use the residue theorem. Take a closed contour in \mathcal{Z}_1 and let this contour approach $\partial\mathcal{Z}_1$. The function $R_\pm(z)\psi_l^\pm(z)\psi_m^\pm(z)$ is continuous on $\{|z| = 1\} \setminus \{z(E_j)\}$ and meromorphic on \mathcal{Z}_1 with simple poles at $z(\rho_j)$ and eventually a pole at $z = 0$.

Due to the properties of \hat{s} , \hat{w} , ψ_l^+ we have

$$R_+ \sim_{z \sim 0} \frac{z^{-(\nu+q)}}{z^{-q}} = z^{-\nu}, \quad \psi_l^+\psi_m^+ \sim_{z \sim 0} z^{l+m}.$$

Suppose $l + m \geq \nu + 1$ (+1 is due to singularity z^{-1} in $d\omega$). Then the integrand is bounded near $z = 0$ and we apply the residue theorem to the only poles at the eigenvalues.

We have ([EMT], (3.23))

$$\frac{dz}{d\lambda} = z \frac{\prod_{j=1}^{q-1}(\lambda - \alpha_j)}{2(\Delta^2(\lambda) - 1)^{1/2}}$$

and if $z_j = z(\rho_j)$ then $\text{Res}_{z=z_j} F(z) = z'(\rho_j) \text{Res}_{\lambda=\rho_j} F(z(\lambda))$.

Then we get

$$F_0^\pm(l, m) = \sum_{j=1}^N \text{Res}_{\rho_j} \left(\frac{R_\pm(z) D^+ D^- \psi_l^\pm(\lambda) \psi_m^\pm(\lambda)}{2(\Delta^2(\lambda) - 1)^{1/2}} \right) = \sum_{j=1}^N \text{Res}_{\rho_j} \left(\frac{D^\mp R_\pm(z) \hat{\psi}_l^\pm(\lambda) \hat{\psi}_m^\pm(\lambda)}{D^\pm 2(\Delta^2(\lambda) - 1)^{1/2}} \right),$$

where we used $\prod_{j=1}^{q-1}(\lambda - \mu_j) = D^+ D^-$. Now we consider F_0^+ only, as calculations for F_0^- are similar. We have

$$F_0^+(l, m) = \sum_{j=1}^N \text{Res}_{\rho_j} \left(\frac{D^-}{D^+} \frac{\hat{s}(z^{-1}) \hat{\psi}_l^+(\lambda) \hat{\psi}_m^+(\lambda)}{\hat{w}(z)(z^q - z^{-q})} \right).$$

The functions $z^{\nu+q} \hat{s}(z^{-1})$, $z^q \hat{w}(z)$ are analytic. Thus

$$z^{\nu+q} \hat{s}(z^{-1}) = z_j^{\nu+q} \hat{s}(z_j^{-1}) + \mathcal{O}(z - z_j), \quad z^q \hat{w}(z) = z_j^q \hat{w}'(\rho_j)(\lambda - \rho_j) + \mathcal{O}(\lambda - \rho_j)^2.$$

The function $\frac{\hat{\psi}_l^+(\lambda) \hat{\psi}_m^+(\lambda)}{(z^q - z^{-q})}$ is bounded at $\rho = \rho_j$ and we write

$$\frac{\hat{s}(z^{-1}) \hat{\psi}_l^+(\lambda) \hat{\psi}_m^+(\lambda)}{\hat{w}(z)(z^q - z^{-q})} = \frac{z^{\nu+q} \hat{s}(z^{-1})}{z^q \hat{w}(z)} \cdot \frac{\hat{\psi}_l^+(\lambda) \hat{\psi}_m^+(\lambda)}{z^q - z^{-q}} \cdot z^{-(\nu+q)+q}$$

We have $\hat{\psi}_l^\pm(z) \sim z^{\pm l}$ as $z \rightarrow 0$. Now if $l + m \geq \nu + 1$ (see above, +1 comes from $d\omega$) then $\hat{\psi}_l^+(\lambda) \hat{\psi}_m^+(\lambda) z^{-\nu-1}$ is bounded. Using that $\hat{s}(z_j^{-1}) = (z_j^q - z_j^{-q})^2 (\hat{s}(z_j))^{-1}$ (due to Lemma 3.2), we get

$$F_0^+(l, m) = \sum_{j=1}^N \frac{D^-(\rho_j)}{D^+(\rho_j)} \frac{(z_j^q - z_j^{-q}) \hat{\psi}_l^+(\rho_j) \hat{\psi}_m^+(\rho_j)}{\hat{w}'(\rho_j) \hat{s}(z_j)} = - \sum_{j=1}^N \gamma_{+,j} \hat{\psi}_l^+(\rho_j) \hat{\psi}_m^+(\rho_j).$$

Then equation (3.15) implies

$$F^+(l, m) = F_0^+(l, m) + \sum_{j=1}^N \gamma_{+,j} \hat{\psi}_l^+(\rho_j) \hat{\psi}_m^+(\rho_j) = 0, \quad l + m \geq \nu + 1,$$

and the Gel'fand-Levitan-Marchenko equation

$$K_+(n, m) + \sum_{l=n}^{+\infty} K_+(n, l) F^+(l, m) = \frac{\delta_{nm}}{K_+(n, n)}, \quad m \geq n,$$

implies that, if $n + m \geq \nu + 1$, the kernel of the transformation operator $K_\pm(n, m)$ satisfies

$$K_+(n, m) = \frac{\delta_{nm}}{K_+(n, n)}, \quad m \geq n, \quad m + n \geq \nu + 1.$$

Thus we get the following properties:

if $2n \geq \nu + 1$, then $K_0(n, n) = \pm 1$; if $n + m \geq \nu + 1$, $m \neq n$, then $K_0(n, m) = 0$.

As by (5.27) in [EMT] we have

$$\frac{a_n}{a_n^0} = \frac{K_+(n+1, n+1)}{K_+(n, n)}, \quad v_n = a_n^0 \frac{K_+(n, n+1)}{K_+(n, n)} - a_{n-1}^0 \frac{K_+(n-1, n)}{K_+(n-1, n-1)},$$

and if $\nu = 2p$, then, $a_n = a_n^0$ for $n \geq p + 1$, and $v_n = 0$ for $n \geq p + 1$. The case $\nu = 2p - 1$ is similar.

Analogously it follows that u_n, v_n are 0 for $n < 0$. \square

Proof of Theorem 1.2. Uniqueness. If $V \equiv (u, v) \in \mathcal{V}_\nu$, then Lemma 3.3 implies that $(\hat{w}, f_0^+) \in \mathfrak{C}_\nu$, which yields a mapping $V \mapsto (f_0^+, \hat{w})$ from \mathcal{V}_ν into \mathfrak{C}_ν .

We will show uniqueness. Let $V \equiv (u, v) \in \mathcal{V}_\nu$. Then Lemma 3.3 gives unique $(\hat{w}, f_0^+) \in \mathfrak{C}_\nu$. Lemma 3.2 yields the norming constants $\gamma_{\pm, j}$, $j = 1, \dots, N$. By 1) in Lemma 3.8 the Hypothesis 1 is satisfied. Then, by Theorem 3.3, these data determine the finitely supported perturbation uniquely. Then we deduce that the mapping $V \mapsto (\hat{w}, f_0^+)$ is an injection.

Surjection of the mapping $V \equiv (u, v) \mapsto (\hat{w}, f_0^+)$ follows from 2) in Lemma 3.8.

3.4 Proof of Theorem 1.3.

First we note that, as the zeros of \hat{w} and $R_- + 1$ are disjoint from the end-points λ_k^\pm , $k = 0, \dots, q$, and from the points μ_j , $j = 1, \dots, q - 1$, such that $\varphi_q(\tilde{\mu}_j) = 0$, then the zeros of

$$R_- + 1 = \frac{w + s}{w} = \frac{2i\Omega f_0^+}{\varphi_q w} = \frac{2i\Omega f_0^+}{a_0^0 \hat{w}}$$

(see (3.9)) coincide with the zeros of f_0^+ . We used also that the zeros of f_0^+ and w are disjoint, which follows from the Wronskian property: as $w = \{f^-, f^+\} = a_0(f_0^- f_1^+ - f_1^- f_0^+)$, then $w(\lambda_0) = 0$, $f_0^+(\lambda_0) = 0$ would imply $f_0^-(\lambda_0) = 0$, which is impossible.

Moreover, the multiplicities of zeros of $R_- + 1$ and f_0^+ coincide, i.e. the zeros are simple.

Now using the zeros of f_0^+ , the polynomial φ_0^+ and the constant c_3 we can reconstruct the polynomial $F(\lambda) = \varphi_q f_0^+(f_0^+)^*$, $\lambda \in \Lambda_1$. We recall that $f^*(\lambda) = \overline{f(\bar{\lambda})}$. The zeros of f_0^+ and the polynomial F were studied in detail in [IK3]. Applying Theorem 1.4 in [IK3] we know that the function f_0^+ is uniquely determined by the polynomials F and φ_0^+ , supposing that the zeros of f_0^+ are simple.

In [IK1] we show that the set of zeros of the polynomial $\mathcal{F} = \hat{w}\hat{w}^*$ on \mathbb{C} coincide with the projection of the set of states of $H : \tilde{\sigma}_{\text{st}}(H)$, with the same multiplicities. Thus, as the polynomial F in [IK3], the polynomial \mathcal{F} can be reconstructed from $\tilde{\sigma}_{\text{st}}(H)$ and the constant $c_3 v_0$ (see (1.21)). Now applying the polynomial interpolation method (see [A]) as in [IK3] we can reconstruct $\hat{w} = 2i\Omega(1 + A) - J$ from $\sigma_{\text{st}}(H)$ using the polynomials A and \mathcal{F} . Now we have also the function $\hat{s} = \frac{2i\Omega}{a_0^0} f_0^+ - \hat{w}$, see (3.9). \square

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